DILLA UNIVERSITY
DEPARTMENT OF MATHEMATICS

Algebra I Exercise 1
due on Dec 14, 2018, 8:30 AM

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1. Let a, b be elements of group G. Show that
   i) $|a| = |a^{-1}|$,
   ii) $|ab| = |ba|$, and
   iii) $|cac^{-1}| = |a|$ for all $c \in G$
2. Prove that the following conditions on a group G are equivalent (**Abelian Relations**):
   (i) G is abelian;
   (ii) $(ab)^2 = a^2b^2$ for all $a, b \in G$;
   (iii) $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$;
   (iv) $(ab)^n = a^n b^n$ for all $n \in \mathbb{Z}$ and all $a, b \in G$;
3. If $a^2 = e$ for all elements a of a group G, then prove that G is abelian. (**Groups of Involutions**)
4. Let Q_8 be the group (under ordinary matrix multiplication) generated by the complex matrices $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, where $i^2 = -1$. Show that Q_8 is a nonabelian group of order 8. The group Q_8 is called the *quaternion group*. [Hint: Observe that $BA = A^3B$, whence every element of Q_8 is of the form A^iB^j. Note also that $A^4 = B^4 = I$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity element of Q_8.]
5. Let S be a nonempty subset of a group G and define a relation on G by $a \sim b$ if and only if $ab^{-1} \in S$. Show that $\sim$ is an equivalence relation if and only if S is a subgroup of G.
6. Let $f : G \to H$ be a homomorphism of groups, A a subgroup of G, and B a subgroup of H. Show that $f^{-1}(B)$ (resp $f(A)$) is a subgroup of G (resp H).